Multidimensional fluid motions with planar waves

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Abstract

In the classical one-dimensional solution of fluid dynamics equations all unknown functions depend only on time t and Cartesian coordinate x. Although fluid spreads in all directions (velocity vector has three components) the whole picture of motion is relatively simple: trajectory of one fluid particle from plane $x = \text{const}$ completely determines motion of the whole plane. Basing on the symmetry analysis of differential equations we propose generalization of this solution allowing movements in different directions of fluid particles belonging to plane $x = \text{const}$. At that, all functions but an angle determining the direction of particle's motion depend on t and x only, whereas the angle depends on all coordinates. In this solution the whole picture of motion superposes from identical trajectories placed under different angles in 3D space. Orientations of the trajectories are restricted by a finite relation possessing functional arbitrariness. The solution describes three-dimensional nonlinear processes and singularities in infinitely conducting plasma, gas or incompressible liquid.

PACS numbers: 47.10.-g, 47.10.A, 47.65.-d

AMS classification scheme numbers: 76W05, 76M60, 35C05, 35N10

Introduction

The widely-used simplification of fluid dynamics equations is an assumption of one-dimensionality of the flow. It is proposed that all unknown functions depend only on two variables: time t and Cartesian coordinate x. Motion of fluid particles is allowed in all directions, however most of interesting processes (waves of compression and rarefaction, strong and weak discontinuities, etc.) take place along one spatial axis Ox. Components of the velocity vector, thermodynamical and all another unknown functions are constant on the planes $x = \text{const}$ and change from one plane to another. This solution is often referred to as fluid motion with planar waves. Being comparatively easy for an analytical analysis, this simplification provides a great deal of information about qualitative properties of fluid motions. However, the classical one-dimensional solution can not describe three-dimensional processes in fluid which in fact might be significant for the correct description of the picture of the flow.

In present work we generalize the described classical one-dimensional solution with planar waves. In our solution velocity vector is decomposed into two components, one of which is parallel and another one is orthogonal to Ox axis. Absolute values (lengthes) of the components and both thermodynamical functions (density and pressure) are supposed to depend only on t and x. This part of solution coincide with the classical one. However,

the angle of rotation of velocity vector about Ox axis is supposed to depend on all independent variables (t, x, y, z). Presence of this function gives the desired generalization of the classical solution.

The proposed representation of the solution was advised by the theory of symmetry analysis of differential equations [1, 2]. Indeed, from the symmetry analysis point of view, the classical one-dimensional solution is an invariant one of rank 2 with respect to the admissible group of shifts along Oy and Oz axis. Whereas the generalized solution is a partially invariant one [1] with respect to the full group of plain isometries consisting of shifts along Oy and Oz axes and rotation about Ox axis.

Class of generalized solutions is happened to be a contansive one. It is described by a closed system of PDEs with two independent variables, which in the special case coincide with classical equations for one-dimensional fluid motions. The angle as a function of four independent variables is determined on solutions of the invariant system from a finite (not differential) relation, which has a functional arbitrariness. The finite relation allows clear geometrical interpretation. This gives opportunity to construct a desired type of fluid motion by choosing appropriate arbitrary function in the functional relation.

Plasma flow governed by the solution possesses a remarkable property. Fluid particles belonging to the same initial plane $x = \text{const}$ at some moment of time circumscribe the same trajectories in 3D space and have identical magnetic field lines attached. However, each trajectory and magnetic field line has its own orientation, which depends on the position of the fluid particle in the initial plane. The orientation is given by the finite relation with functional arbitrariness. Thus, with the same shape of trajectories and magnetic field lines one can construct infinitely many pictures of fluid motions by varying admissibly directions of particles spreading.

Intensively studied in recent scientific literature solution of ideal compressible or incompressible fluid equations which is called “singular vortex” or “Ovsyannikov vortex” [3, 4, 5, 6, 7, 8, 9, 10] can be treated as the analogous generalization of one-dimensional motion with spherical waves. In this solution absolute values of the tangential and normal to spheres $r = \text{const}$ components of velocity vector field depend only on time t and distance r to the origin. An angle of rotation of the vector field about the radial direction Or is a function on all independent variables. This solution also allows symmetry interpretation as the partially invariant one with respect to the admissible group of sphere isometries, i.e. of rotations in $\mathbb{R}^3$.

The generalized one-dimensional solution with planar waves for ideal gas dynamics equations was first obtained in [13]. For all we known, it was not analyzed in details for its physical content. In present work we observe equations of ideal magnetohydrodynamics. Cases of ideal gas dynamics and ideal liquid can be obtained in limits of zero magnetic field $\mathbf{H} \equiv 0$ and constant density $\rho = \text{const}$ respectively.

The paper is organized as follows. We start from the formulation of the representation of solution, which is prescribed by symmetry properties of the main model of ideal magnetohydrodynamics. Substitution of the representation of the solution into the system of equations brings a highly-overdetermined system of PDEs for the non-invariant function — angle of rotation of the vector fields about Ox axis. Investigation of the overdetermined system reveals two main cases, when some auxiliary function h is equal or not equal to

zero. From the mechanical point of view these two cases correspond to the compressible or incompressible (divergence-free) vector field which is obtained as a projection of the velocity field into Oyz plane. In both cases the overdetermined system is reduced to some compatible invariant subsystem of PDEs with two independent variables and a finite implicit relation for the non-invariant function. We give geometrical interpretation of the finite relation, which allows keeping track of the singularities, which may take place in the flow. We prove that particles trajectories and magnetic field lines are planar curves. Moreover, these curves are the same for all particles, which start from the same initial plane $x = \text{const}$. This gives opportunity to construct a pattern of the trajectory and magnetic field line. The complete 3D picture of the flow is obtained by attaching the pattern to every point in fixed Oyz plane in accordance to the directional field defined by the finite relation for the non-invariant function. Remarkable, that the same pattern of magnetic line and trajectory attached to different directional field in Oyz plane produces variety of pictures of plasma motion in 3D space. As an example, the solution is used for explicit description of the plasma flow in axisymmetric canal with curved conducting walls.

1 Representation of the solution and preliminary analysis

1.1 Representation of the solution

The system of ideal magnetohydrodynamics (tension comes to pressure, thermal conductivity is zero, electric conductivity is infinite) has the form [17]

$$D\,\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \tag{1.1}$$

$$D\,\mathbf{u} + \rho^{-1}\nabla p + \rho^{-1}\mathbf{H}\times\mathrm{rot}\,\mathbf{H} = 0, \tag{1.2}$$

$$D\,p + A(p,\rho)\,\mathrm{div}\,\mathbf{u} = 0, \tag{1.3}$$

$$D\,\mathbf{H} + \mathbf{H}\,\mathrm{div}\,\mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{u} = 0, \tag{1.4}$$

$$\mathrm{div}\,\mathbf{H} = 0, \quad D = \partial_t + \mathbf{u}\cdot\nabla. \tag{1.5}$$

Here $\mathbf{u} = (u, v, w)$ is the fluid velocity vector, $\mathbf{H} = (H, K, L)$ is the magnetic vector field; p and ρ are pressure and density. The state equation $p = F(S, \rho)$ with the entropy S gives rise to function $A(p, \rho)$ defined by $A = \rho\,(\partial F/\partial\rho)$. All unknown functions depend on time t and Cartesian coordinates $\mathbf{x} = (x, y, z)$.

In the case of arbitrary state equation $p = F(S, \rho)$ equations (1.1)–(1.5) admit 11-dimensional Lie group G_{11} of point transformations, which is 10-dimensional Galilean group extended by the homothety [11, 12]. Optimal system of subgroups ΘG_{11} was constructed in [15, 16], see also [14]. Examination of ΘG_{11} shows, that the partially invariant solution of described type is generated by 3-dimensional subgroup $G_{3.13} \subset G_{11}$ with Lie algebra $L_{3.13}$ spanned by the infinitesimal generators $\{\partial_y,\ \partial_z,\ z\partial_y - y\partial_z + w\partial_v - v\partial_w + L\partial_K - K\partial_L\}$ (we use the subgroups numeration according to [14]).

Indeed, Lie group $G_{3.13}$ is spanned by shifts along Oy and Oz axes and simultaneous rotations about the first coordinate axis in $\mathbb{R}^3(\mathbf{x})$, $\mathbb{R}^3(\mathbf{u})$, and $\mathbb{R}^3(\mathbf{H})$. Invariants of this group of transformations in the space of independent variables and dependent functions

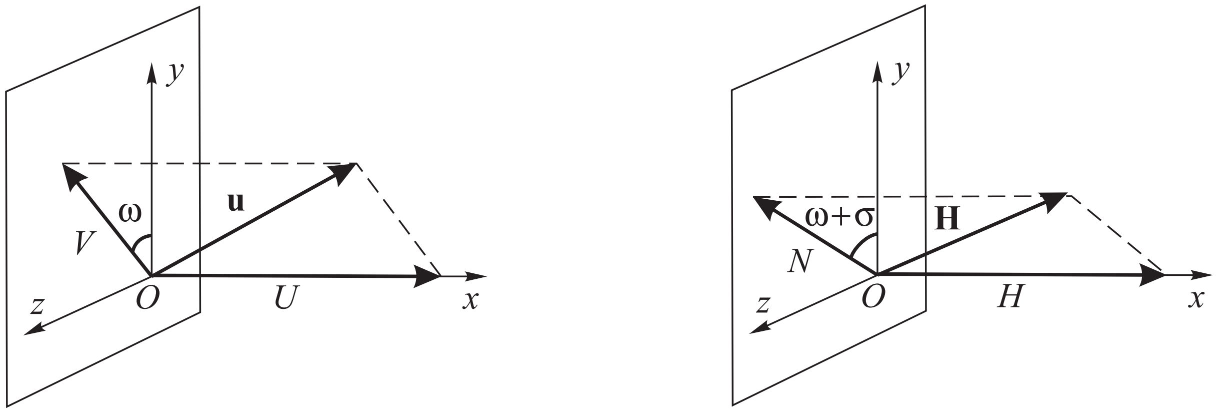

Figure 1: Representation of velocity vector $\mathbf{u}$ and magnetic field vector $\mathbf{H}$ in the partially invariant solution. All functions but ω depend on t and x, whereas $\omega = \omega(t, x, y, z)$.

$\mathbb{R}^4(t, \mathbf{x}) \times \mathbb{R}^8(\mathbf{u}, \mathbf{H}, p, \rho)$ are

$$t, \quad x, \quad u, \quad V = \sqrt{v^2 + w^2}, \quad p, \quad \rho, \quad H, \quad N = \sqrt{K^2 + L^2}, \quad \text{and} \quad vK + wL. \tag{1.6}$$

The last invariant may be treated as angle σ between the projections of vectors $\mathbf{u}$ and $\mathbf{H}$ into Oyz plane (see figure 1). The general theory of partially invariant solutions may be found in [1]. The representation of partially invariant solution is obtained by assigning a functional dependence between the group invariants (1.6). In particular, for the solution of rank 2 (two invariant independent variables) and defect 1 (one non-invariant function) it gives the following representation of solution:

$$\begin{aligned}
&u = U(t, x), && H = H(t, x),\\
&v = V(t, x)\cos\omega(t, x, y, z), && K = N(t, x)\cos\big(\omega(t, x, y, z) + \sigma(t, x)\big),\\
&w = V(t, x)\sin\omega(t, x, y, z), && L = N(t, x)\sin\big(\omega(t, x, y, z) + \sigma(t, x)\big),\\
&p = p(t, x), \quad \rho = \rho(t, x), \quad S = S(t, x).
\end{aligned} \tag{1.7}$$

Here only the non-invariant function $\omega(t, x, y, z)$ depends on all original independent variables. Functions U, V, H, N, σ, p, ρ are invariant with respect to $G_{3.13}$. They depend only on invariant variables t and x. These functions will be referred to as invariant ones. The system of equations for determination of invariant and non-invariant functions will be called the submodel of the main model of ideal magnetohydrodynamics.

1.2 Analysis of the submodel

Substitution of the representation (1.7) into (1.1)–(1.5) gives the following result. The continuity equation (1.1) allows introduction of new unknown invariant function $h(t, x)$, defined by the following relation

$$\widetilde{D}\,\rho + \rho(U_x + hV) = 0. \tag{1.8}$$

Hereinafter $\widetilde{D}$ denotes the invariant part of the differentiation along the trajectory

$$\widetilde{D} = \partial_t + U\partial_x.$$

The remaining part of the continuity equation gives an equation for function ω:

$$\sin\omega\,\omega_y - \cos\omega\,\omega_z + h = 0. \tag{1.9}$$

Another equations for invariant functions follow from the first components of momentum (1.2) and induction equations (1.4), and also pressure equation (1.3).

$$\widetilde{D}\,U + \rho^{-1}p_x + \rho^{-1}NN_x = 0, \tag{1.10}$$
$$\widetilde{D}\,H + hHV = 0, \tag{1.11}$$
$$\widetilde{D}\,p + A(p,\rho)(U_x + hV) = 0. \tag{1.12}$$

The rest of system (1.1)–(1.5) gives rise to the overdetermined system for function ω. From a nondegenerate linear combination of equations (1.2) in projections to Oy and Oz axes one obtains

$$\begin{gathered}\rho V\omega_t + \big(\rho UV - HN\cos\sigma\big)\,\omega_x + \big(\rho V^2\cos\omega - N^2\cos\sigma\cos(\omega+\sigma)\big)\,\omega_y \\ +\big(\rho V^2\sin\omega - N^2\cos\sigma\sin(\omega+\sigma)\big)\,\omega_z - H(N_x\sin\sigma + N\cos\sigma\sigma_x) = 0.\end{gathered} \tag{1.13}$$

$$\begin{gathered}HN\sin\sigma\,\omega_x + N^2\sin\sigma\cos(\omega+\sigma)\,\omega_y + N^2\sin\sigma\sin(\omega+\sigma)\,\omega_z \\ +\rho\,\widetilde{D}V + HN\sin\sigma\,\sigma_x - HN_x\cos\sigma = 0.\end{gathered} \tag{1.14}$$

The same operation with remaining two induction equations (1.4) provides

$$\begin{gathered}N\omega_t + (NU - HV\cos\sigma)\,\omega_x + VN\sin\sigma\sin(\omega+\sigma)\,\omega_y \\ -VN\sin\sigma\cos(\omega+\sigma)\,\omega_z + N\widetilde{D}\sigma + HV_x\sin\sigma = 0.\end{gathered} \tag{1.15}$$

$$\begin{gathered}HV\sin\sigma\,\omega_x + NV\cos\sigma\sin(\omega+\sigma)\,\omega_y \\ -NV\cos\sigma\cos(\omega+\sigma)\,\omega_z - \widetilde{D}N + HV_x\cos\sigma - NU_x = 0.\end{gathered} \tag{1.16}$$

Finally, equation (1.5) is transformed to

$$N\big(\sin(\omega+\sigma)\omega_y - \cos(\omega+\sigma)\,\omega_z\big) - H_x = 0. \tag{1.17}$$

The overdetermined system (1.9), (1.13)–(1.17) for non-invariant function ω should be investigated for compatibility [18]. At that we observe only solution with functional arbitrariness in determination of function ω. This condition, in particular, implies non-reducibility of the solution to the classical one-dimensional solution with planar waves.

Function ω determines with only constant arbitrariness if it is possible to express all first-order derivatives of ω from the system of equations (1.9), (1.13)–(1.17). In order to prohibit this situation one should calculate a matrix of coefficients of the derivatives of function ω and vanish all its rank minors. This leads to the following four cases:

$$1.\ H = 0;\quad 2.\ N = 0;\quad 3.\ V = 0;\quad 4.\ \sigma = 0\ \text{ or } \sigma = \pi. \tag{1.18}$$

By definition (1.7) functions V and N are non-negative. Values $\sigma = \pi$ and $\sigma = 0$ in the case 4 (1.18) differ only by the sign of function N. Both can be observed in the same framework for $\sigma = 0$, non-negative V and arbitrary N.

Cases 2 and 3 in classification (1.18) correspond to the magnetic field or velocity parallel to Ox-axis. Both of them embed into the case $\sigma = 0$. Indeed, if $\sigma = 0$ then the velocity vector at each particle and its magnetic field vector belong to the plane, which is orthogonal to Oyz coordinate plane. Therefore, cases 2 and 3 are degenerate versions of this more general situation. Case 4 will be observed as the main case in the following calculations. In case of pure gas dynamics $\mathbf{H} \equiv 0$ three of four conditions (1.18) satisfied automatically, hence the solution is irreducible without any additional restrictions.

1.3 Case of planar magnetic field

Let us first observe the case $H = 0$, when the magnetic field vector is parallel to Oyz plane. The compatibility condition of equations (1.9) and (1.17) in this case is

$$\big(\cos(\omega+\sigma)\,\omega_y + \sin(\omega+\sigma)\,\omega_z\big)h = 0. \tag{1.19}$$

For $h = 0$ the determinant of the homogenous system of algebraic equations (1.9), (1.17) for ω_y and ω_z is $\sin\sigma$. Hence, the solution is non-trivial only for $\sigma = 0$ or $\sigma = \pi$. The case $h \neq 0$ leads to the reduction following from equations (1.17) and (1.19). Thus, the non-trivial solution exists only for $\sin\sigma = 0$, i.e. case 1 in the classification (1.18) contains in case 4.

2 The main case $h \neq 0$

2.1 Equations of the submodel

Let us observe the main case $\sigma = 0$. From the mechanical point of view it corresponds to a plasma flow where velocity and magnetic field vectors at each particle are coplanar to Ox axis. Equations (1.8), (1.10)–(3.36) belong to the invariant part of the submodel. Besides, equation (1.14) gives

$$\widetilde{D}\,V - \rho^{-1}HN_x = 0. \tag{2.20}$$

From equation (1.16) taking into account (1.9) one obtains

$$\widetilde{D}\,N + NU_x - HV_x + hNV = 0. \tag{2.21}$$

Finally, equation (1.17) due to the relation (1.9) can be written as

$$H_x + hN = 0. \tag{2.22}$$

In addition to the equation (1.9), the non-invariant part of the determining system contains two equations, which follow from (1.13), (1.15):

$$\rho V\omega_t + \big(\rho\,UV - HN\big)\,\omega_x + \big(\rho V^2 - N^2\big)(\cos\omega\,\omega_y + \sin\omega\,\omega_z) = 0, \tag{2.23}$$
$$N\omega_t + (NU - HV)\,\omega_x = 0. \tag{2.24}$$

Elimination of the derivative ω_t from equations (2.23), (2.24) gives the following classifying relation

$$(\rho V^2 - N^2)\big(H\omega_x + N(\cos\omega\,\omega_y + \sin\omega\,\omega_z)\big) = 0. \tag{2.25}$$

We observe only the case when the second factor in (2.25) vanishes. The compatibility conditions of equations (1.9), (2.24), and (2.25) are

$$N\widetilde{D}\,h - HVh_x = 0, \tag{2.26}$$

$$Hh_x + h^2 N = 0. \tag{2.27}$$

For $h \neq 0$ there is an integral

$$H = H_0 h, \quad H_0 = \text{const}. \tag{2.28}$$

Thus, the submodel's equations are reduced to the following ones.

$$\widetilde{D}\,\rho + \rho(U_x + hV) = 0. \tag{2.29}$$

$$\widetilde{D}\,U + \rho^{-1}p_x + \rho^{-1}NN_x = 0. \tag{2.30}$$

$$\widetilde{D}\,V - \rho^{-1}H_0 hN_x = 0, \tag{2.31}$$

$$\widetilde{D}\,p + A(p,\rho)(U_x + hV) = 0, \tag{2.32}$$

$$\widetilde{D}\,N + NU_x - H_0 hV_x + hNV = 0, \tag{2.33}$$

$$\widetilde{D}\,h + Vh^2 = 0, \quad H_0 h_x + hN = 0. \tag{2.34}$$

The obtained system (2.29)–(2.34) inherits the overdetermination of the initial MHD equations (1.1)–(1.5). However, its compatibility conditions satisfied by virtue of the system itself. Indeed, the only nontrivial compatibility condition of the system (2.29)–(2.34) is given by two equations (2.34) for function h. Cross-differentiation of (2.34) shows that their compatibility condition coincide with equation (2.33), i.e. is already contained in the system. The most general Cauchy problem for system (2.29)–(2.34) requires assigning functions ρ, U, V, p, N at $t = 0$ as functions of x, and fixing a constant value of h at $t = 0$ over some plane $x = \text{const}$. For pure gas dynamics $\mathbf{H} \equiv 0$ the second equation (2.34) satisfies identically, hence the initial data for h become $h(0, x) = h_0(x)$. System (2.29)–(2.34) equipped by the suitable initial data can be solved numerically. It also allows exact reductions to systems of ODEs since the admitted symmetry group is obviously nontrivial.

Equations (2.23)–(2.25) for the non-invariant function can be integrated. Function ω determines by the following implicit equation

$$F(y - \tau\cos\omega,\ z - \tau\sin\omega) = 0 \tag{2.35}$$

with $\tau = 1/h$ and arbitrary smooth function F. In case of pure gas dynamics $\mathbf{H} \equiv 0$ equation (2.24) identically satisfied. Therefore, function F in the general formula (2.35) for function ω also arbitrarily depends on ξ: $\xi_t + U\xi_x = 0$. Results of the performed calculations are summarized in the following theorem.

Theorem 2. *In the main case $\sigma = 0$ and $h \neq 0$ the invariant functions are determined by the system of differential equations (2.29)–(2.34). The non-invariant function ω is given by the implicit equation (2.35) with arbitrary smooth function F.*

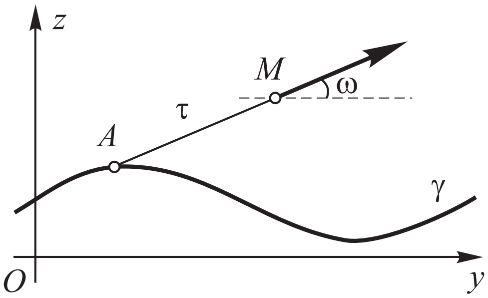

Figure 2: Geometric interpretation of the solution $\omega = \omega\big(\tau(t,x), y, z\big)$ of the implicit equation (2.35). Curve $\gamma : \ F(y,z) = 0$ is determined by the same function F as in (2.35). Function ω at given point M is the angle between the direction of line segment AM and Oy axis, where $A \in \gamma$ and $|AM| = \tau$.

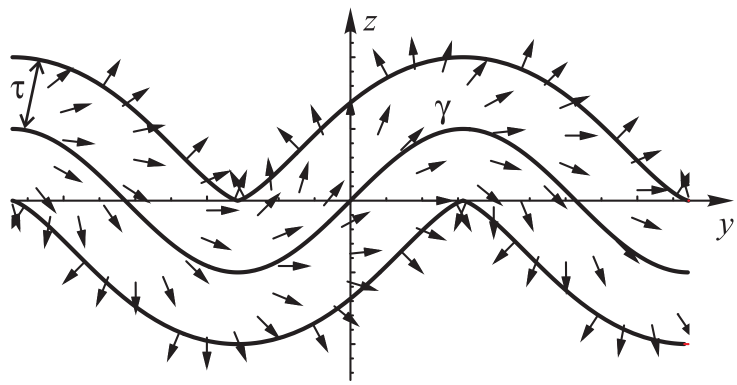

Figure 3: The field of directions is defined by the implicit equation (2.35) in the stripe of determinacy of width 2τ with curve $\gamma : \ F(y,z) = 0$ as a medial line. In this example $F = z - \sin y$. At the points of limiting equidistants the field of directions is orthogonal to the equidistants.

2.2 Geometrical construction of the field of directions

Here we give an algorithm for solving the implicit relation (2.35) over some fixed plane $x = x_0$ at time $t = t_0$. Suppose that function F in (2.35) is fixed. This specifies a curve $\gamma = \{(y,z) \,|\, F(y,z) = 0\}$. In order to find angle ω at arbitrary point $M = (y,z)$ one should draw a line segment AM of the length τ such that $A \in \gamma$. The direction of AM gives the required angle ω as it is shown in figure 2. Function ω is only defined at points located within distance τ from the curve γ. The rest of Oyz plane does not belong to the domain of ω. Boundaries of the domain of ω are τ-equidistants to γ. As x grows, function τ changes according to the solution of equations (2.29)–(2.34). This prescribes modification of the ω-domain over different planes $x = \mathrm{const}$. Thus, the domain of function ω (hence, of the whole solution (1.7)) over each plane $x = \mathrm{const}$ is a stripe of determinacy of the width 2τ with curve γ as a centerline (see figure 3). The stripe of determinacy is bounded by equidistants curves to γ. Over the boundaries of ω-domain the field of directions ω is orthogonal to the boundaries.

Inside its domain function ω is multiply-defined. Indeed, there are could be several line segments AM with $A \in \gamma$ giving rise to several branches of function ω. However, it is always possible to choose a single-valued and continuous branch of ω.

Discontinuities of ω may appear in cases when the equidistants to γ have the dovetail singularities. The observations illustrated by figure 4 show that every branch of function ω necessary have a line of discontinuity inside or at the border of the dovetail. In figure 4 the curve γ is a sinusoid shown at the bottom of figures; the curve on the top is the equidistant shifted at large enough distance τ. For the convenience we draw the circle of radius τ with center at chosen point M. Each intersection of the circle with γ gives rise to a branch of ω. Let us take M outside of the dovetail (figure *a*). There are two branches of ω at M. As M moves towards the borders of the dovetail, both branches change continuously (figure *b*). At the border of the dovetail the new branch of ω appears (figure *c*). The latter splits into two branches inside the dovetail (figure *d*). As M reaches the right boundary of the dovetail the two "old" branches of ω sticks together (figure *e*) and disappear as M leaves the dovetail (figure *f*). One can chase, that the branches of ω obtained on the right-hand side of the dovetail are different from the ones existed on the left-hand side of the dovetail.

The dovetails do not appear if $\tau < \min\limits_{\mathbf{x}\in\gamma} R(\mathbf{x})$, where $R(\mathbf{x})$ is a curvature radius of curve γ at $\mathbf{x}$. So, one can avoid the singularities either by choosing the solution with small enough τ or by fixing the curve γ with large curvature radius. Described discontinuities takes the solution out of class (1.7). They can not be interpreted in shock waves framework. Indeed, over the line of discontinuity only the direction of the magnetic and velocity vector fields change, while their absolute values together with thermodynamics functions remain continuous. Another type of transverse or alfvéic waves [17, 19] characteristic to ideal MHD equations also can not explain the discontinuity as long as the magnetic and velocity fields rotates not across the front of discontinuity.

Appearance of the dovetail singularities physically mean magnetic field lines, which pass through different point in some initial plane $x = \text{const}$ collide in their further development. This happens if the function τ increases along the magnetic lines such that the τ-equidistants to γ became non-smooth. In the vicinity of the collision point the solution leaves the prescribed class (1.7); the corresponding fluid flow should be observed either in general 3D framework, or in terms of an extended main model, i.e. taking into account magnetic or kinematic viscosity as it is observed in magnetic reconnection problems [20]. This nonlinear process is specific to the constructed solution, and can not take place in the classical one-dimensional solution with planar waves, where all magnetic lines are parallel to each other.

3 Case $h = 0$

3.1 Equations of the submodel

From the mechanical point of view this case means that the projection of vector field $\mathbf{u}$ into the plane $x = \text{const}$ is incompressible, i.e. its divergence is zero. This case is observed separately because the non-invariant function ω is determined by different algorithm.

For $h = 0$ integral (2.28) is not valid. Instead, equations (1.11) and (2.22) give

$$H = H_0 = \text{const}.$$

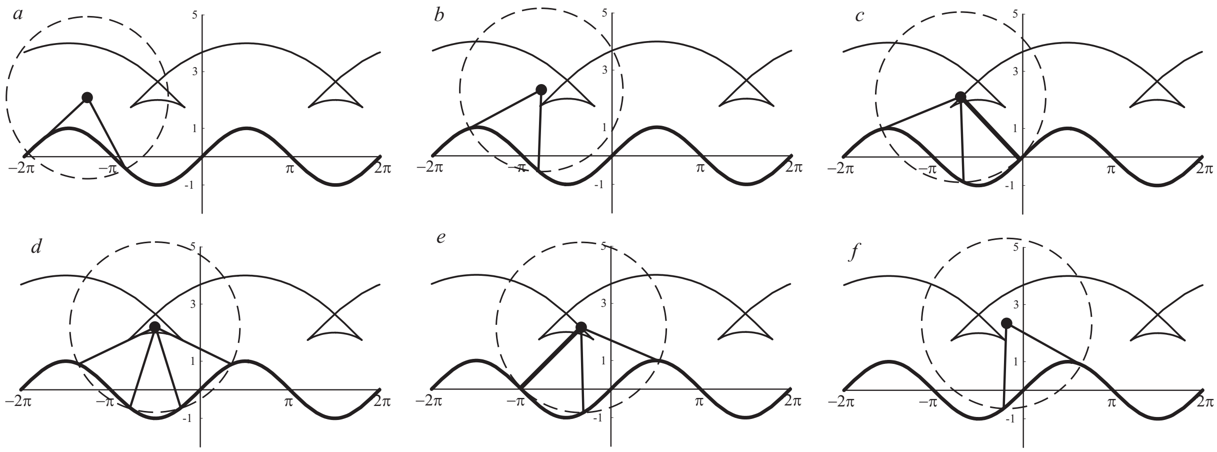

Figure 4: The behaviour of function ω over the dovetail. There are two branches of ω outside the dovetail in Figures (a), (b) and (f); three branches of ω at the borders of the dovetail in Figures (c) and (e); and four branches of solution inside the dovetail in Figure (d).

Thus, equations of the invariant system are

$$\begin{aligned} &\widetilde{D}\,\rho + \rho\, U_x = 0,\\ &\widetilde{D}\,U + \rho^{-1} p_x + \rho^{-1} N N_x = 0,\\ &\widetilde{D}\,V - \rho^{-1} H_0 N_x = 0,\\ &\widetilde{D}\,p + A(p,\rho)\, U_x = 0,\\ &\widetilde{D}\,N + N U_x - H_0 V_x = 0. \end{aligned} \tag{3.36}$$

This system of 5 equations serves for determination of 5 unknown functions U, V, N, p, and ρ. The non-invariant function ω is restricted by equations (1.9), (2.24), and (2.25). Suppose that its solution $\omega = \omega(t,x,y,z)$ for $N \neq 0$ and $\rho V^2 - N^2 \neq 0$ is given implicitly by the equation $\Phi(t,x,y,z,\omega) = 0$, $\Phi_\omega \neq 0$. The system (1.9), (2.24), and (2.25) transforms as follows

$$\Phi_k = 0, \quad \Phi_t + U\Phi_x + V\Phi_j = 0, \quad H_0\,\Phi_x + N\Phi_j = 0. \tag{3.37}$$

Here Ojk is a Cartesian frame of reference rotated on angle ω about the origin.

$$j = y\cos\omega + z\sin\omega, \quad k = -y\sin\omega + z\cos\omega. \tag{3.38}$$

Integrals of system (3.37) are ω and $j - \varphi(t,x)$, where function $\varphi(t,x)$ satisfies the overdetermined system

$$\varphi_t + U\varphi_x = V, \quad H_0\,\varphi_x = N. \tag{3.39}$$

The compatibility condition of equations (3.39) is the last equation of the invariant system (3.36). Differential one-form

$$H_0 d\varphi = (H_0 V - NU)dt + N dx$$

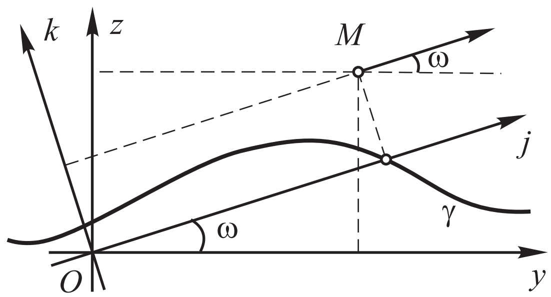

Figure 5: Given a value of ω at some point M, the auxiliary Ojk frame of reference is defined as shown. The projection of M into the Oj axis is called the base point for M. The set of all the base points for different M with different $\omega(M)$ forms the basic curve γ.

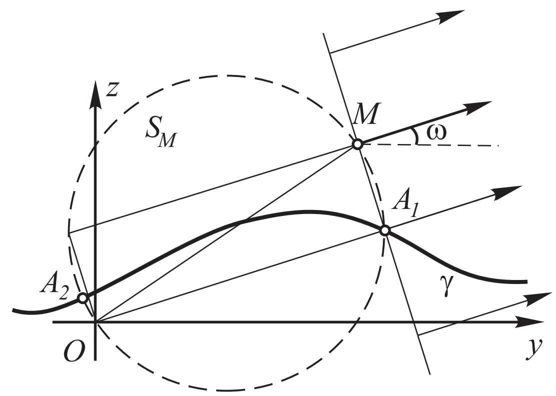

Figure 6: Given curve γ one can find ω at any point M of the corresponding Oyz plane. Circle S_M with diameter OM should be drawn. Let A_i be points of intersection of S_M with γ. For each A_i the angle ω at M is given by the direction OA_i as shown.

is closed, therefore function φ can be found by integration as

$$\varphi(t,x) = \int\limits_{(t_0,x_0)}^{(t,x)} d\varphi.$$

Note, that the initial data for function φ is given by only one constant $\varphi(t_0, x_0)$. The non-invariant function ω can be taken in the form of the finite implicit equation

$$j = f(\omega) + \varphi(t,x) \tag{3.40}$$

with arbitrary smooth function f. The result is formulated in the following theorem.

Theorem 3. *In the case $\sigma = h = 0$ the invariant functions are determined from equations (3.36), (3.39). Function ω is given by the implicit equation (3.40).*

3.2 Construction and properties of the field of directions

Now we clarify a geometrical interpretation of the implicit relation (3.40). Let us fix a plane $x = x_0$ and time $t = t_0$. For simplicity we assume $\varphi(t_0, x_0) = 0$. Let the value of ω satisfying (3.40) is known at some point $M = (y, z)$ of the plane $x = x_0$. Consider a Cartesian frame of reference Ojk turned counterclockwise on angle ω with respect to Oyz (see figure 5). By the construction, j-coordinate of point M and angle ω are related by $j = f(\omega)$. All points with the same coordinate j and arbitrary coordinate k satisfy the same relation. A point satisfying the relation (3.40) with zero coordinate k will be referred to as the base point for chosen values of j and ω. The locus of all base points for various j and ω gives the basic curve γ. On the plane Oyz the basic curve γ is defined in polar coordinates $y = r\cos\theta$, $z = r\sin\theta$ by the equation $r = f(\theta)$. Note, that since the value of j can have arbitrary sign, both positive and negative values of polar coordinate r are allowed in the construction of γ.

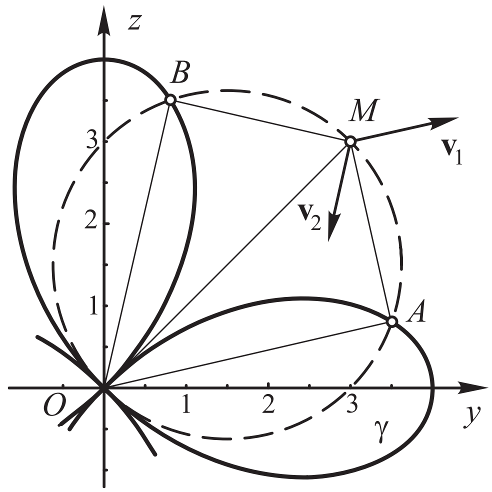

Figure 7: Curve a γ is defined by equation $r = \cos 2\theta$ where both positive and negative values of r are allowed. Point B corresponds to the part of the curve with negative r. The direction $\mathbf{v}_2$ assigned to B is therefore opposite to the one given by the segment OB.

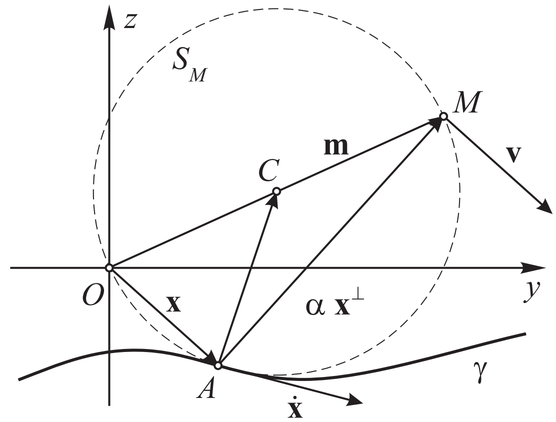

Figure 8: Point M belongs to the boundary of the domain of function ω if the circle S_M is tangent to γ at some point A. From the elementary geometry vectors OA and AM are orthogonal. This allows expressing vector $\mathbf{m}$ in terms of $\mathbf{x}$ and $\mathbf{x}^{\perp}$.

The obtained geometrical interpretation provides an algorithm of construction of the vector field, which is defined by the angle ω of deviation from the positive direction of the Oy axis. Angle ω is determined from the solutions of implicit equation (3.40). Suppose, that function f in equation (3.40) is given. This means, that one can construct the basic curve γ by the formula $r = f(\theta)$ in polar frame of reference on Oyz plane. Determination of angle ω at the point $M = (y, z)$ of the plane $x = x_0$ requires the following operations as illustrated in figure 6.

1. Draw a circle S_M with diameter OM.
2. Find the intersection points A_i of the circle S_M with curve γ. If S_M does not intersect γ then M does not belong to the domain of ω.
3. The angle between the line segment OA_i and a positive direction of Ox axis gives a value of the angle ω at point M (see figure 6).
4. Angle ω has the same value at all points of the line passing through the line segment A_iM.

As mentioned before, function f can be both positive and negative. Negative f corresponds to negative coordinate j. Thus, if the point of intersection of the curve γ and auxiliary circle S_M belongs to those part of the curve, which corresponds to the negative values of f, then the vector field should be taken with the negative sign, i.e. instead of ω one should take $\omega + \pi$. This situation is shown in figure 7. Here the curve γ is determined by the equation $r = \cos 2\theta$. The circle with diameter OM for $M = (3, 3)$ has two points of intersection with curve γ. Point A belongs to the "positive" part of curve

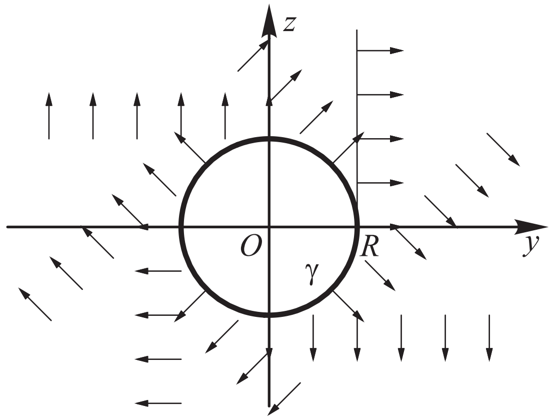

Figure 9: The vector field defined by the curve $\gamma : y^2 + z^2 = R^2$.

γ, therefore it defines the direction $\mathbf{v}_1$, codirectional with the segment OA. Point B lies on the "negative" part of γ, i.e. the corresponding direction $\mathbf{v}_2$, is opposite to the one, defined by the segment OB.

Next, it is necessary to find the domain of function $\omega = \omega(t_0, x_0, y, z)$ defined by the implicit equation (3.40) over the plane $x = x_0$. Assume that curve γ is given. Point M belongs to the boundary of the domain if the circle S_M with diameter OM is tangent to curve γ at some point A (see figure 8). Let the position vector of point M be $\mathbf{m}$. Parametrization of γ is taken in the form $\mathbf{x} = \mathbf{x}(s)$ with some parameter $s \in \Delta \subset \mathbb{R}$. From the elementary geometry $\mathbf{m} = \mathbf{x} + \alpha\mathbf{x}^{\perp}$, where $\mathbf{x}^{\perp} \cdot \mathbf{x} = 0$. The tangency condition of the circle and curve γ gives $(\mathbf{m}/2 - \mathbf{x}) \cdot \dot{\mathbf{x}} = 0$. Here and further the upper dot denotes the differentiation with respect to s. Substitution of the expression for $\mathbf{m}$ form the first equality into the second one provides $(\alpha\mathbf{x}^{\perp}/2 - \mathbf{x}/2) \cdot \dot{\mathbf{x}} = 0$. The scalar α is then determined by

$$\alpha = \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{\mathbf{x}^{\perp} \cdot \dot{\mathbf{x}}}.$$

Thus, the border of the domain of function ω has the following parametrization

$$\mathbf{m} = \mathbf{x} + \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{\mathbf{x}^{\perp} \cdot \dot{\mathbf{x}}}\mathbf{x}^{\perp}. \quad \mathbf{x} = \mathbf{x}(s), \quad s \in \Delta \subset \mathbb{R}. \tag{3.41}$$

Note, that $\mathbf{m}$ does not depend on the choice of the sign and length of $\mathbf{x}^{\perp}$. At the border's points the vector field defined by ω has $\mathbf{x}$ direction. This direction is orthogonal to the border. Indeed,

$$\dot{\mathbf{m}} \cdot \mathbf{x} = (\dot{\mathbf{x}} + \dot{\alpha}\mathbf{x}^{\perp} + \alpha\dot{\mathbf{x}}^{\perp}) \cdot \mathbf{x} = \dot{\mathbf{x}} \cdot \mathbf{x} + \frac{\mathbf{x} \cdot \dot{\mathbf{x}}}{\mathbf{x}^{\perp} \cdot \dot{\mathbf{x}}}\dot{\mathbf{x}}^{\perp} \cdot \mathbf{x} = 0.$$

The last expression vanishes because from $\mathbf{x} \cdot \mathbf{x}^{\perp} = 0$ it follows $\dot{\mathbf{x}} \cdot \mathbf{x}^{\perp} = -\mathbf{x} \cdot \dot{\mathbf{x}}^{\perp}$.

As an example, let us take γ to be the circle $y^2 + z^2 = R^2$. The border of the domain of ω in this case coincide with the circle γ because for each point $\mathbf{x}$ of the border one has $\mathbf{x} \cdot \dot{\mathbf{x}} = 0$. The corresponding vector field describes a flow from the cylindrical source and shown in figure 9. In limit $R = 0$ one obtains a vector field corresponding to the rotation around the origin.

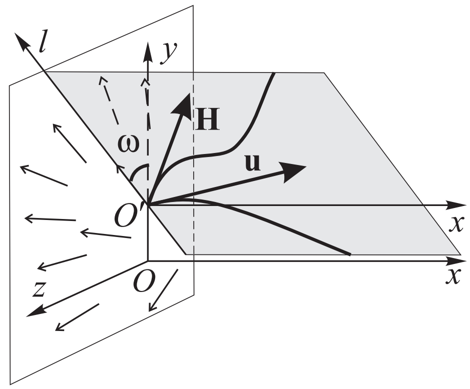

Figure 10: Trajectories and magnetic field lines are planar curves, which are the same for all particles, belonging to the same plane $x = \text{const}$. In order to determine the flow in the whole space it is required to set up an admissible vector field of directions in some plane $x = x_0$ (i.e. to determine function ω consistent with equations (2.35) or (3.40)) and calculate trajectory and magnetic field line for arbitrary particle in this plane. The whole picture of the flow is obtained by attaching the trajectory and the magnetic line pattern to each point on the plane $x = x_0$ in accordance with the vector field of directions.

4 Particles trajectories and magnetic field lines

4.1 Trajectory and magnetic field line pattern

First of all, let us notice that from equations (2.23), (2.24) for $\rho V^2 - N^2 \neq 0$ follows the equality

$$D\omega = 0. \tag{4.42}$$

The trajectory of each particle is a planar curve. Indeed, equation (4.42) implies that angle ω has constant value along each trajectory. Hence, the whole trajectory belongs to the plane, which is parallel to Ox axis and turned on angle ω about this axis. The same holds for a magnetic field line, because vanishing of the second factor in (2.25) is equivalent to constancy of ω along each magnetic curve. Thus, for each particle its trajectory and magnetic field line are planar curves, which lie in the same plane defined by the angle ω.

The second important property follows from the representation of the solution (1.7). Let us set up a Cauchy problems for trajectory of some particle. The particle moves in its plane, hence in this plane the motion is completely defined by components of velocity U and V. These two functions depend only on invariant variables t and x. Therefore, for any two particles, which belong to the same plane $x = x_0$ at initial time $t = t_0$ the Cauchy problems for the trajectories coincide. Of course, the two different particles move in their own planes, but both trajectories as planar curves are exactly the same. Similar observation is true for any two magnetic lines passing through two different points in the same plane $x = x_0$. Thus, one can construct a pattern by calculating the trajectory and the magnetic field line for any particle in the plane $x = x_0$. The pattern attached to each points in the plane $x = x_0$ inside of the domain of function ω according to the field of directions defined by function ω produces the 3D picture of trajectories and magnetic field lines in the whole space. The described algorithm is illustrated in figure 10.

In order to construct the pattern let us observe a plane of motion of some particle, which is located at initial time $t = t_0$ at some point $M = (x_0, y_0, z_0)$. This plane is parallel to Ox axis and turned about Ox axis on angle ω. Cartesian frame of reference is defined in the plane of motion as follows. The origin O' of the frame is placed at the projection of point M into Oyz plane. One of the coordinate axes is chosen to be parallel to Ox axis and denotes by the same letter x. Another axis $O'l$ is placed orthogonally to $O'x$ such that the frame $O'xl$ has right orientation (see figure 10). Particle's trajectory in this frame of reference is defined by the solution of the Cauchy problem

$$\frac{dx}{dt} = U(t, x), \quad x(t_0) = x_0. \tag{4.43}$$

The dependence $x = x(t, x_0)$ given by a solution of (4.43) allows finding the dependence $l = l(t)$ along the trajectory by the formula

$$l(t) = \int_{t_0}^{t} V(t, x(t, x_0))dt. \tag{4.44}$$

The planar curve determined by the dependencies $x = x(t, x_0)$ and $l = l(t)$ forms a pattern of the trajectory for any particle, which belongs to the plane $x = x_0$ at $t = t_0$. Equations of particle's trajectory in initial $Oxyz$-frame are restored in the form

$$x = x(t, x_0), \quad y = y_0 + l(t)\cos\omega_0, \quad z = z_0 + l(t)\sin\omega_0. \tag{4.45}$$

Here $\omega_0 = \omega(t_0, \mathbf{x}_0)$ is the value of angle ω taken at initial time $t = t_0$ at point M.

The magnetic field line at $t = t_0$ is an integral curve of the magnetic vector field. The pattern of the magnetic curve passing at $t = t_0$ through the plane $x = x_0$ is given by

$$l(x) = \int_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)}ds.$$

Equations of the magnetic field curve in $Oxyz$ frame of reference are restored as

$$y = y_0 + \cos\omega_0 \int_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)}ds, \quad z = z_0 + \sin\omega_0 \int_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)}ds. \tag{4.46}$$

Derivation of these formulae is similar to those given for the trajectory (4.45).

Thus, the following properties of plasma motion holds (see figure 10).

- Trajectories and magnetic lines lie entirely in planes, which are orthogonal to the Oyz-plane and turned on angle ω about Ox axis.
- All particles, which belong at some moment of time $t = t_0$ to a plane $x = x_0$, circumscribe the same trajectories in planes of each particle motion. Magnetic field lines passing through a plane $x = x_0$ are also the same planar curves.
- Angle of rotation about Ox-axis of the plane containing the trajectory and the magnetic line of each particle is given by function ω, which satisfies equation (2.35) or (3.40).

4.2 Domain of the solution in 3D space

The constructions above show that the whole area in 3D space occupied by the solution is obtained as follows. In fixed plane $x = x_0$ function ω has some (in many cases, finite) definition domain, bounded by τ-equidistants to γ for $h \neq 0$ and by the curve (3.41) for $h = 0$. In both cases the field of direction defined by ω in $x = x_0$ plane is orthogonal to the boundary of the ω-domain. In order to obtain boundaries of the whole 3D domain of the solution one should attach the magnetic line pattern, calculated on some particular solution of the invariant system, to every point of the boundary of ω-domain in plane $x = x_0$ according to the usual algorithm. This gives a canal woven from magnetic lines which pass through boundaries of the $\omega(t,x_0, y, z)$-domain and intersect $x = x_0$ plane. The walls of the canal can be interpreted as rigid infinitely conducting pistons. Due to the well-known property of magnetic field line freezing-in, the walls are impermeable for plasma. In case of stationary solution the walls are fixed. In non-stationary case the walls extend or shrink according to the behavior of function τ for $h \neq 0$ and φ for $h = 0$. In case of finite ω-domain (it can always be restricted to a finite one) each x-cross-section of the 3D-domain of the solution is finite, therefore both magnetic and kinetic energy have finite value in each x-layer.

4.3 Stationary flow

As an example we observe a stationary solution of system (2.29)–(2.34). Suppose that all unknown functions depend on x only. This leads to the following system of ODEs:

$$U\rho' + \rho(U' + hV) = 0. \tag{4.47}$$

$$UU' + \rho^{-1}p' + \rho^{-1}NN' = 0. \tag{4.48}$$

$$UV' - \rho^{-1}H_0hN' = 0, \tag{4.49}$$

$$Up' + A(p, \rho)(U' + hV) = 0, \tag{4.50}$$

$$UN' + NU' - H_0hV' + hNV = 0, \tag{4.51}$$

$$Uh' + Vh^2 = 0, \quad H_0h' + hN = 0. \tag{4.52}$$

Elimination of the derivative h' in equations (4.52) gives the finite relation

$$H_0Vh = UN, \tag{4.53}$$

which states collinearity of the magnetic and velocity fields at each particle. The same property holds for the analogous spherical solution [9]. Equation (4.51) is satisfied identically by virtue of (4.53).

Equation (4.50) gives entropy conservation

$$S = S_0. \tag{4.54}$$

Equation (4.47) under condition (4.53) gives the flow rate integral

$$\rho U = nh, \quad n = \text{const}. \tag{4.55}$$

Substitution of the obtained integrals into (4.49) allows finding the following relation between the tangential components of velocity and magnetic fields

$$nV - H_0N = m, \quad m = \text{const}. \tag{4.56}$$

Integration of equation (4.48) gives the Bernoulli integral

$$U^2 + V^2 + 2\int \frac{dp}{\rho} = b^2, \quad b = \text{const.} \tag{4.57}$$

The only equation left to integrate is any of two equations (4.52). With its aid all unknown functions may be expressed in terms of the "potential" $\tau = 1/h$ as

$$U = \frac{m\tau + H_0^2\tau'}{n\tau\tau'}, \quad V = \frac{m\tau + H_0^2\tau'}{n\tau}, \quad H = \frac{H_0}{\tau}, \quad N = \frac{H_0\tau'}{\tau}, \quad \rho = \frac{n^2\tau'}{m\tau + H_0^2\tau'}. \tag{4.58}$$

a) Let $m \neq 0$. Using the admissible dilatations it is convenient to make $m = n = \operatorname{sign}(\tau\tau')$. Expressions (4.58) become

$$U = \frac{\tau + H_0^2\tau'}{\tau\tau'}, \quad V = \frac{\tau + H_0^2\tau'}{\tau}, \quad H = \frac{H_0}{\tau}, \quad N = \frac{H_0\tau'}{\tau}, \quad \rho = \frac{\tau'}{\tau + H_0^2\tau'}. \tag{4.59}$$

Substitution of (4.59) into the Bernoulli integral (4.57) produces an equation for τ. In case of polytropic gas with the state equation $p = S\rho^\gamma$ it has the following form

$$\left(\frac{\tau + H_0^2\tau'}{\tau\tau'}\right)^2 + \left(\frac{\tau + H_0^2\tau'}{\tau}\right)^2 + \frac{2\gamma S_0}{\gamma - 1}\left(\frac{\tau'}{\tau + H_0^2\tau'}\right)^{\gamma-1} = b^2. \tag{4.60}$$

This ODE for $\tau(x)$ is not resolved with respect to the derivative τ', which complicates its investigation. Examples of analysis of such non-resolved ODEs can be found in [4]–[7]. One can show that there are several branches of solution $\tau(x)$ of equation (4.60) passing through each point in (x, τ) plane, which correspond to different relations between the velocity U and the characteristics speeds of MHD system (1.1)–(1.5). It is possible to switch between different branches of the solution via fast or slow shock waves. However, this investigation lies outside of the scope of this paper.

b) In case $m = 0$ after some straightforward simplifications we obtain the following solution of system (2.29)–(2.34):

$$\begin{gathered} U = H_0^2 \operatorname{sech} x, \quad V = H_0^2 \tanh x, \quad \tau = \cosh x, \\ H = H_0 \operatorname{sech} x, \quad N = H_0 \tanh x, \quad \rho = H_0^{-2}, \quad S = S_0. \end{gathered} \tag{4.61}$$

One can check that (4.61) represents a special case of the more general S. Chandrasekhar solution [21]. This solution is also invariant with respect to infinite group of Bogoyavlenskij transformations [22]. The simplicity of solution (4.61) gives opportunity to use it for demonstration of geometrical algorithms given in previous sections.

Streamlines and magnetic field lines coincide and are given by formulas (4.45) with $x_0 = 0$ and

$$l(x) = \cosh x - 1. \tag{4.62}$$

In each plane of particle's motion the streamline is a half of catenary. Note, that solution (4.61) can be continuously adjoined with the uniform flow along Ox axis. Indeed, in section $x = 0$ all functions in (4.61) and their derivatives take values compatible with

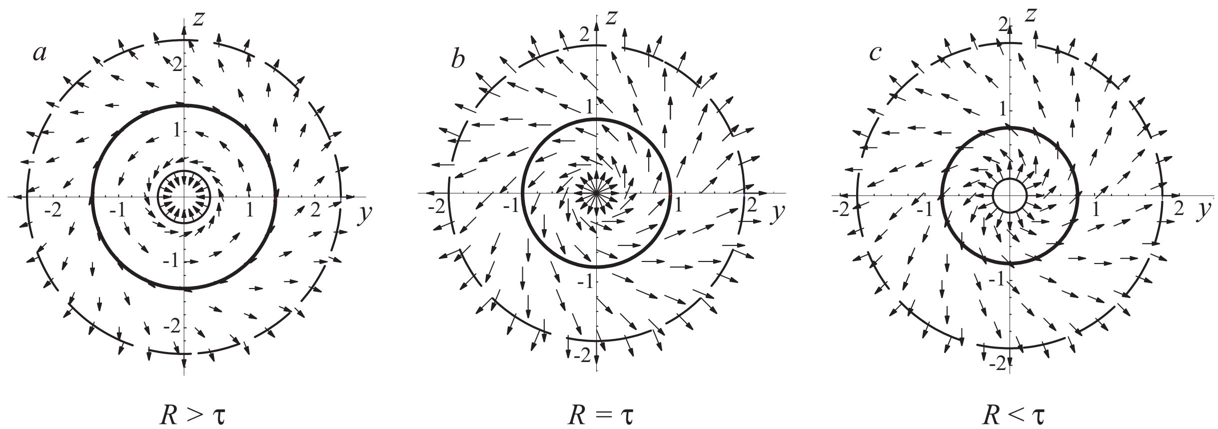

Figure 11: Field of direction obtained by the algorithm of section 2.2. Here γ is the middle circle of radius R. Three cases according to the relation between R and τ distinguishes. In all cases the domain of the vector fields is an annular stripe of determinacy between two equidistant curves (inner and outer circles in the diagrams).

the uniform flow. Let us construct a solution, which switches the uniform flow to the generalized one-dimensional solution (4.61) at the section $x = 0$. The corresponding streamline is a straight lines for $x < 0$ and a half of the catenary for $x \geq 0$. In order to get the whole three-dimensional picture of motion this streamline pattern should be attached to each point of the plane $x = 0$ according to the direction field defined by function ω.

Function ω is determined by the implicit equation (2.35). Algorithm of section 2.2 requires assigning some particular function F, or some curve $\gamma : F(y, z) = 0$. Let the curve γ be a circle $y^2 + z^2 = R^2$. The corresponding function ω is determined at each point of the plane $x = 0$ by equation (2.35). Figure 11 shows the vector fields obtained for different relations between τ and R. For $R > \tau$ the vector field is defined in the annular area between two circles of radii $R \pm \tau$. On the inner equidistant circle $|\mathbf{x}| = R - \tau$ the vector field is directed outside of the stripe of determinacy towards the origin. In case $R = \tau$ the inner equidistant circle shrinks into the origin $\mathbf{x} = 0$. At that, the vector field becomes multiply-determined at this point. Finally, for $R < \tau$ the inner equidistant turns inside out and becomes a circle of radius $\tau - R$ with the vector field on it directed inside of the stripe of the determinacy. These three vector fields generate different pictures of motion in whole 3D space.

The streamline pattern described above should be attached to each points of Oyz plane inside the corresponding domain of ω according to the directional fields shown in figure 11. Because of the obvious central symmetry of the vector fields the whole picture of motion is axially-symmetrical. The axial section of the area in 3D space, occupied by the corresponding flow is shown in figure 12.

We assume that uniform flow for $x < 0$ changes at section $x = 0$ to the flow, described by the solution (4.61). Depending on the relation between $\tau(0)$ and R three different pictures of motion are possible. Each particle moves along the same planar curve, however orientation of the streamlines in the space differ from one particle to another. Three-dimensional visualization of the motion for $R > \tau(0)$ is shown in figure 13.

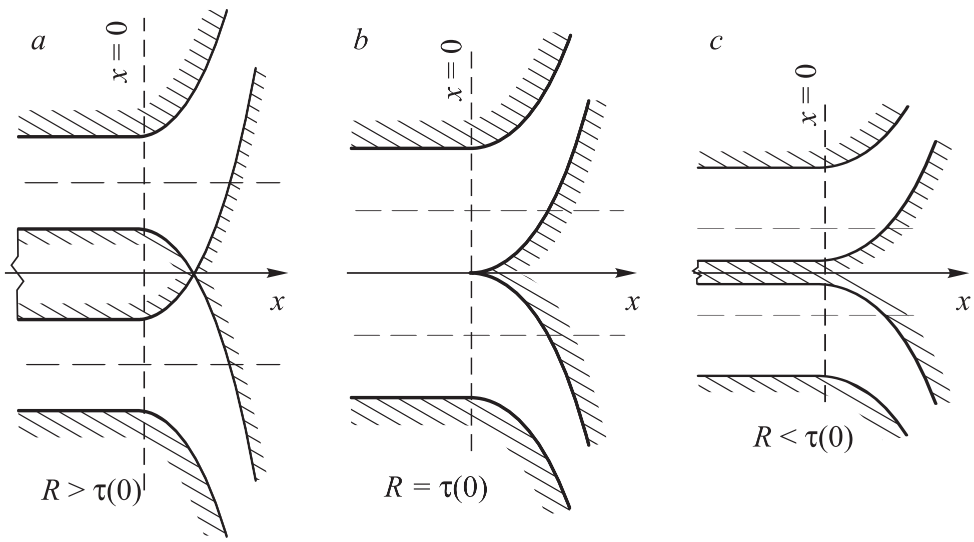

Figure 12: Axial sections of axially-symmetrical canal occupied by the plasma flows. The uniform flow in cylindrical canal for $x < 0$ switches at section $x = 0$ to the flow in the curvilinear canal for $x > 0$ described by the solution (4.61). The boundary of the canal is a rigid wall. Cases a, b and c correspond to the vector fields in figure 11. In the diagrams a and c the canal has an inner cylindrical core.

Conclusion

In present work a new solution of ideal fluid dynamics equations, describing three-dimensional motions of plasma, gas and liquid is constructed. The solution is determined by a system of equations with two independent variables, which is analogous to the classical system for one-dimensional fluid motions. At that, the new solution describes spatial nonlinear processes and singularities, which are impossible to obtain in the classical framework.

In the constructed solution particles trajectories and magnetic field lines are flat curves. Trajectory of each curve and its magnetic field line belong to the same plane parallel to Ox axis. In contrast to the classical one-dimensional solution, plane of motion of each particle has its own orientation, which is given by an additional finite relation. The functional arbitrariness of the finite relation allows varying the geometry of obtained motion in accordance to the problem under consideration. Depending on the chosen geometry, singularities on the border of the region, occupied by fluid, may appear. In such cases particles may collide at the border of the domain of the flow. The criterion of singularities appearance in terms of invariant properties of the arbitrary function, which specifies the geometry of the flow is given.

The obtained solution may be used as a test for numerical modeling of complicated three-dimensional flows of infinitely conducting plasma. It also may serve for theoretical investigations of three-dimensional singularities of the ideal fluid and plasma motions.

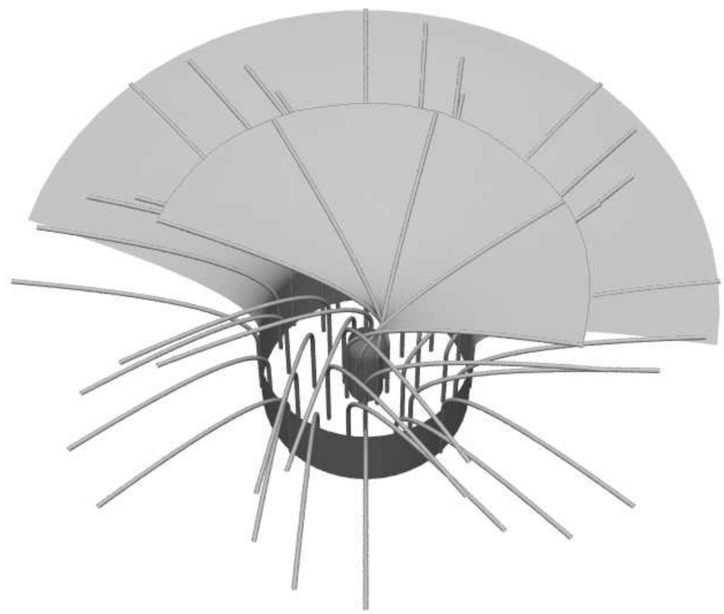

Figure 13: Tree-dimensional visualization of motion. Fragments of the canal's walls and the streamlines are shown. Each streamline has a shape of the same flat curve. Orientation of each streamline is defined by the vector field in figure 11a. The axial section of the canal is represented in figure 12a.

Acknowledgements

Author would like to thank professor O.I. Bogoyavlenskij and Mathematical & Statistical Department of Queen's University for hospitality and inspiring atmosphere, which stimulated this research. Author also acknowledge the support of Russian Foundation for Basic Research (project 05-01-00080), of President Programme of Support of the Leading Scientific Schools (grant Sc.Sch.-5245.2006.1), and of Integration Project 2.15 of Siberian Branch of RAS.

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